IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Supergroups have no extra topology

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1983 J. Phys. A: Math. Gen. 16 3457

(http://iopscience.iop.org/0305-4470/16/15/011)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 16:51

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 16 (1983) 3457–3462. Printed in Great Britain

Supergroups have no extra topology

R F Picken
Institut für Theorie der Elementarteilchen, Freie Universität Berlin, FB20, WE4, Arnimallee 14, D-1000 Berlin 33, West Germany

Received 6 April 1983

Abstract. The global topology of supergroups is studied and it is found that they are equivalent as far as homotopy is concerned to the product of the underlying ordinary Lie groups.

1. Introduction

Supergroups have for a long time been familiar objects in supergravity theory (see van Niewenhuizen 1981 for a general review). However, most authors have considered only their local, linear, superalgebraic properties. In view of the importance of global, in particular homotopic, properties of Lie groups in gauge theories (for instance the importance of the relation $\pi_3(\mathrm{SU}(2))=\mathbb{Z}$ for instanton physics), an investigation of the global nature of supergroups is a worthwhile undertaking. Our result is that supergroups exhibit no new topological features beyond the homotopy groups of the underlying Lie groups.

Our material is organised as follows. In § 2 we introduce Grassman algebras. In § 3 we define supermatrices and supergroups. In § 4 as an example we consider the topology of $\mathrm{OSp}(1|2;\mathbb{R})$. In § 5 we prove our general result concerning supergroups. Section 6 contains comments and conclusions.

2. Grassman algebras

We wish to extend the ordinary real number system to a system of commuting and anticommuting numbers. This is best described in terms of the Grassman or exterior algebra $\bigwedge E$ of a real n-dimensional vector space E. Let e_i, $i=1,\ldots,n$ be a basis of E; then $\bigwedge E$ is spanned by the set of 2^n basis vectors $(1,\ e_i,\ e_i\wedge e_j,\ e_i\wedge e_j\wedge e_k,\ldots,\ e_1\wedge e_2\wedge\ldots\wedge e_n)$ where i, j, k etc run from 1 to n such that $i<j<k<\ldots$ etc and $\wedge$ denotes the totally antisymmetric wedge product. As well as the vector space structure on $\bigwedge E$ the $\wedge$ product of elements of $\bigwedge E$ is defined by linearity from the $\wedge$ product of basis vectors. An example occurring in mathematical physics is the exterior algebra of forms at a point x of the n-dimensional manifold M, $\bigwedge T^*M_x$ spanned by $(1,\ \mathrm{d}x^i,\ \mathrm{d}x^i\wedge\mathrm{d}x^j,\ldots,\ \mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n)$ with the usual wedge product of forms.

We denote the Grassman algebra $\bigwedge\mathbb{R}^n$ by B_n. Consider the elements of B_n spanned by basis vectors consisting of (the wedge product of) precisely d basis vectors of $\mathbb{R}^n$. This subspace is denoted $B_{n,d}$ and elements of $B_{n,d}$ are said to have degree d. If $d>n$

© 1983 The Institute of Physics

we define $B_{n,d}=0$. B_n also naturally decomposes into an even and an odd part denoted B_n^0 and B_n^1 respectively spanned by basis vectors of even resp odd degree. The $\wedge$ product respects both the $\mathbb{Z}$ grading of degree and the $\mathbb{Z}_2$ grading of evenness/oddness in the sense that

$$B_{n,d_1}\wedge B_{n,d_2}\subset B_{n,d_1+d_2}, \qquad B_n^i\wedge B_n^j\subset B_n^{(i+j)\bmod 2}.$$

Note that elements of B_n^0 commute among themselves and with elements of B_n^1, whereas elements of B_n^1 anticommute among themselves.

All we have said can straightforwardly be generalised to complex Grassman algebras of complex vector spaces.

3. Supermatrices and supergroups

Consider the vector space consisting of column vectors

$$(a_1,\ldots,a_m,\theta_1,\ldots,\theta_p)^{\mathrm{T}} \qquad \text{with } a_i\in B_n^0\ \forall i,\ \theta_j\in B_n^1\ \forall j.$$

Linear transformations on this space are described by supermatrices $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ where the entries of A $(m\times m)$ and D $(p\times p)$ are in B_n^0 and the entries of B $(m\times p)$ and C $(p\times m)$ are in B_n^1. In order to have the identity

$$\left[\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x_i\\ \theta_j\end{pmatrix}\right]^{\mathrm{T}}=(x_i\quad \theta_j)\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{\mathrm{T}}$$

we must define

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}A^{\mathrm{T}} & C^{\mathrm{T}}\\ -B^{\mathrm{T}} & D^{\mathrm{T}}\end{pmatrix}$$

because of the anticommutativity of the elements of B and θ_j, which are interchanged in the transposition. Note that we have omitted to write the $\wedge$ product explicitly.

Consider now a non-degenerate bilinear form on this vector space described by the *real* $(m+p)\times(m+p)$ matrix

$$\begin{pmatrix}P & 0\\ 0 & Q\end{pmatrix}=J$$

with P $m\times m$ symmetric and Q $p\times p$ antisymmetric. The linear transformations preserving this form are supermatrices $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ satisfying

$$\begin{pmatrix}A^{\mathrm{T}} & C^{\mathrm{T}}\\ -B^{\mathrm{T}} & D^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}P & 0\\ 0 & Q\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}P & 0\\ 0 & Q\end{pmatrix} \tag{1}$$

and such matrices form a (generally continuous) group called a super (Lie) group. By considering elements infinitesimally close to the identity

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{pmatrix}$$

we obtain the super (Lie) algebra associated with this supergroup, i.e. Gothic matrices satisfying

$$\begin{pmatrix}P & 0\\ 0 & Q\end{pmatrix}\begin{pmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{pmatrix}+\begin{pmatrix}\mathfrak{A}^{\mathrm{T}} & \mathfrak{C}^{\mathrm{T}}\\ -\mathfrak{B}^{\mathrm{T}} & \mathfrak{D}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}P & 0\\ 0 & Q\end{pmatrix}=0.$$

Most work in this area has concerned the description and classification of these superalgebras. Our interest concerns the topology of the set of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfying (1), being the topology inherited as a subspace of the total parameter space $\mathbb{R}^d$, $d = 2^{n-1}(m+p)^2$.

4. An example: the topology of OSp(1|2; ℝ)

If

$$P = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \qquad (r \text{ times } 1, s \text{ times } -1)$$

and

$$Q = \text{block diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \qquad (t \text{ times})$$

then the supergroup preserving J is known as $\mathrm{OSp}(r, s|2t; \mathbb{R})$ or $\mathrm{OSp}(r|2t; \mathbb{R})$ if $s = 0$. The supergroup $\mathrm{OSp}(1|2; \mathbb{R})$ consists of matrices satisfying

$$\begin{pmatrix} a & \delta & \kappa \\ -\beta & e & l \\ -\gamma & f & m \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a & \beta & \gamma \\ \delta & e & f \\ \kappa & l & m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

leading to the quadratic conditions

$$\begin{aligned} &a^2 - 2\kappa\delta = 1, && a\beta - e\kappa + l\delta = 0, \\ &a\gamma - f\kappa + m\delta = 0, && -\beta\gamma - lf + em = 1. \end{aligned} \tag{2}$$

Note that we have denoted elements of B_n^0 resp B_n^1 with Roman resp Greek letters.

We choose as our Grassman algebra B_2 with basis $\{1, e^1, e^2, e^1 \wedge e^2 \equiv e^1 e^2\}$ and display the components of even and odd elements in the following manner:

$$a = a_0 + a_{12} e^1 e^2, \qquad \beta = \beta_1 e^1 + \beta_2 e^2. \tag{3}$$

The equations (2) now split according to degree:

$$a_0^2 = 1, \qquad -l_0 f_0 + e_0 m_0 = 1, \tag{4}$$

$$\left.\begin{aligned} a_0\beta_i - e_0\kappa_i + l_0\delta_i = 0 \\ a_0\gamma_i - f_0\kappa_i + m_0\delta_i = 0 \end{aligned}\right\} i = 1, 2, \tag{5}$$

$$\begin{aligned} &2a_0 a_{12} - 2(\kappa_1\delta_2 - \kappa_2\delta_1) = 0, \\ &-(\beta_1\gamma_2 - \beta_2\gamma_1) - (l_0 f_{12} + l_{12} f_0) + (e_0 m_{12} + e_{12} m_0) = 0. \end{aligned} \tag{6}$$

The inhomogeneous equations (4) describe the parameter spaces of O(1) and Sp(2; ℝ) respectively. O(1) consists of two points $a_0 = \pm 1$ and Sp(2; ℝ) is given, under a change of basis

$$\begin{aligned} &l_0 = y_1 + y_2, && e_0 = y_3 + y_4, \\ &f_0 = y_1 - y_2, && m_0 = y_3 - y_4, \end{aligned} \tag{7}$$

by the quadric in $\mathbb{R}^4$

$$y_2^2 + y_3^2 = 1 + y_1^2 + y_4^2$$

which has topology $S^1 \times \mathbb{R}^2$ as $y_2^1 + y_3^2 \geq 1$ (and hence has a non-trivial fundamental group π_1).

Choosing a particular solution of (4), the remaining homogeneous equations (5) and (6) yield nothing of topological interest. Equations (5) just describe hyperplanes in the remaining parameter space $\mathbb{R}^{13}$ and it is fairly easy to convince oneself that the hyperboloidal hypersurfaces (6) contain nothing of homotopic interest. For instance, choosing $a_0 = 1$, we have, under a similar change of basis to (7), the hypersurface

$$z_2^2 + z_3^2 = a_{12} + z_1^2 + z_4^2$$

which is topologically trivial as a_{12} is independently variable and in particular can take the value zero.

5. The topological triviality of supergroups

We will call a supergroup preserving the form J topologically trivial if it is homotopically equivalent to $G \times H$, where G and H are ordinary Lie groups preserving P and Q respectively. ('Homotopically equivalent' means that the supergroup and $G \times H$ have the same homotopy groups π_i ($i = 0, 1, \ldots$).) In § 4 our reasoning amounted to the statement that $\mathrm{OSp}(1|2; \mathbb{R})$ is topologically trivial. We now prove the following general result.

Theorem 1. All supergroups (of the type described in § 3) are topologically trivial.

Proof. Let G_s be a supergroup. We will prove the homotopic equivalence of G_s and $G \times H$ by showing that $G \times H$ is a strong deformation retract of G_s. ($A \subset X$ is a strong deformation retract of X if there exists a *continuous* map $f: X \to A$ such that:

(1) f restricted to A is the identity map on A;

(2) there exists a continuous one-parameter family of maps f_t: $X \to A$, $t \in [0, 1]$ such that $f_0(x) = x$, $f_1(x) = f(x)$ and $f_t(a) = f_0(a)$, $\forall t \in [0, 1]$, $\forall a \in A$. We say that f is homotopic to id_x relative to A.

As an example the circle $x^2 + y^2 = 1$, $z = 0$ is a strong deformation retract of the cylinder $x^2 + y^2 = 1$, $-1 \leq z \leq 1$. In this case f is the projection map onto the (x, y) plane and the continuous family of maps f_t progressively shrinks the height of the cylinder to zero.) G_s is an intersection of quadrics in the total parameter space $\mathbb{R}^d$ where d is given by $(m+p)^2\, 2^{n-1}$ if we choose the Grassman algebra B_n. From (1), G_s is given by solutions of

$$\begin{pmatrix} A^{\mathrm{T}}PA + C^{\mathrm{T}}QC & A^{\mathrm{T}}PB + C^{\mathrm{T}}QD \\ -[A^{\mathrm{T}}PB + C^{\mathrm{T}}QD]^{\mathrm{T}} & -B^{\mathrm{T}}PB + D^{\mathrm{T}}QD \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where we used the properties $P^{\mathrm{T}} = P$, $Q^{\mathrm{T}} = -Q$.

Now, given a solution, we obtain a new solution by scaling each real parameter with a factor λ^d ($d \in \{0, 1, \ldots, n\}$) where d is the degree of the Grassman element parametrised. For instance in (3) we replace a and β by

$$a_\lambda = a_0 + \lambda^2 a_{12} e^1 e^2, \qquad \beta_\lambda = \lambda \beta_1 e^1 + \lambda \beta_2 e^2.$$

This is indeed a new solution as the inhomogeneous equations of degree zero

$$A_0^{\mathrm{T}} P A_0 = P, \qquad D_0^{\mathrm{T}} Q D_0 = Q, \tag{8}$$

are unaffected by the scaling, whilst the homogeneous equations corresponding to degree $d>0$ are simply multiplied by a factor λ^d using the underlying $\mathbb{Z}$ grading of B_n, discussed in §2.

If we denote this mapping of G_s to itself by g_λ, then it is easily seen that the set of solutions of (8) i.e. $G\times H$ is a strong deformation retract of G_s using the continuous one-parameter family of maps g_{1-t}, $t\in[0, 1]$.

6. Comments and conclusion

The extension of the previous result to complex Grassman algebras is straightforward. However, it is not clear to the author what the relation is between these supergroups and the supergroups known as $\mathrm{SU}(r, s|t; \mathbb{C})$ which have unit superdeterminant (see the next comment) and supposedly preserve the sesquilinear form

$$\bar{z}^{\mathrm{T}}Pz+\bar{\varphi}^{\mathrm{T}}Q\varphi=z^{+}Pz+\varphi^{+}Q\varphi$$

where z_i $(i=1,\ldots m=r+s)$, φ_j $(j=1,\ldots t)$ are even resp odd elements of a complex Grassman algebra, P is $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ (r times 1, s times -1) and Q is $\mathrm{diag}(1,\ldots,1)$ (t times 1). The quadratic matrix relation satisfied by an element of the supergroup is

$$\begin{pmatrix} A^{+}PA+C^{+}QC & A^{+}PB+C^{+}QD \\ (-A^{+}PB+C^{+}QD)^{+} & -B^{+}PB+D^{+}QD\end{pmatrix}=\begin{pmatrix} P & 0 \\ 0 & Q\end{pmatrix}$$

which implies $B=C=0$. Hence in our language these would be trivial as supergroups having anticommuting part zero.

As with ordinary Lie groups we may require our supergroup matrices to be 'special', meaning that the superdeterminant or 'Berezinian' equals 1 where

$$\mathrm{sdet}\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\det A(\det(D-CA^{-1}B))^{-1}=\det(A-BD^{-1}C)(\det D)^{-1} \qquad (9)$$

and $\mathrm{sdet}(M_1M_2)=(\mathrm{sdet}\,M_1)(\mathrm{sdet}\,M_2)$.

The two determinant factors in (9) can be calculated using the same expression as for an ordinary determinant because they contain commuting entries only. Furthermore $\det D$ must be an invertible even Grassman algebra element which implies that $(\det D)_0\neq 0$.

We observe that the condition $\mathrm{sdet}\,M=1$ is preserved under the rescaling introduced in the proof of theorem 1. Hence a special supergroup G_s is homotopically equivalent to the ordinary Lie group $G\times_{\mathrm{R}}H$ where the restricted product $\times_{\mathrm{R}}$ means that we only consider pairs (g, h) with $\det g=\det h$. Topologically $G\times_{\mathrm{R}}H$ is $SG\times SH\times S^1$ (where $SG=\{g\in G|\det g=1\}$) for complex groups and $SG\times SH\times S^0$ for real groups. (S^0, the zero sphere, consists of two points, corresponding to the two components $\det g=\det h=1$, $\det g=\det h=-1$.)

We wish to emphasise, however, that as with ordinary Lie groups the additional condition of speciality destroys rather than creates topology. For instance, an ordinary complex supergroup is homotopically equivalent to $G\times H=SG\times SH\times S^1\times S^1$, and speciality annihilates one of the S^1 factors.

This result can probably be related to work by Czyż (1981) who considers vector bundles with transition functions in $\mathrm{GL}(k, \bigwedge V)$ with V an auxiliary vector bundle and shows that smooth bundles of this type reduce to GL(k, $\mathbb{R}$) or GL(k, $\mathbb{C}$) bundles. (For holomorphic bundles, however, this superextension is non-trivial.)

The question arises whether these supergroups are super Lie groups in the rigorous sense of Rogers (1980, 1981). Roughly speaking this means that the supergroup locally looks like $B_n^{p,q} = B_n^0 \times \ldots \times B_n^0 \times B_n^1 \times \ldots B_n^1$ (p resp q times) and that the transition functions and group operations can be written as power series with coefficients in B_n. Returning to the first equation defining OSp(1|2; $\mathbb{R}$) (2) and choosing the Grassman algebra B_2, this equation defines a four-dimensional subset of the six-dimensional parameter space. However, it is not even clear what type of supermanifold this might produce ($(p, q) = (2, 0)$, $(1, 1)$ or $(0, 2)$), starting as we do from one commuting and two anticommuting variables. It would be interesting to investigate this point further.

In conclusion we have studied the topology of supergroups, being a class of matrix groups with Grassman algebra-valued entries, and have found that they exhibit no new homotopic features beyond the homotopy of the underlying ordinary Lie groups.

Acknowledgment

I am most grateful to the Royal Society for financial support in the form of a European Exchange Fellowship.

References

Czyż J 1981 *Lecture Notes in Mathematics* **838** 110–25
van Nieuwenhuizen P 1981 *Phys. Rep.* **68C** 189–398
Rogers A 1980 *J. Math. Phys.* **21** 1352–65
—— 1981 *J. Math. Phys.* **22** 939–45